

# SOME RESULTS ON THE VANISHING CONJECTURE OF DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

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ABSTRACT. In this paper we prove four cases for the vanishing conjecture of differential operators with constant coefficients and also a conjecture on the Laurent polynomials with no holomorphic parts, which were proposed in [Zh3] by the third author. We will also give two counter-examples to show that both the vanishing conjecture and Duistermaat and van der Kallen's theorem [DK] fail in the setting of (Laurent) formal power series.

## 1. Introduction

Let  $z = (z_1, z_2, \dots, z_n)$  be  $n$  commutative free variables and  $\mathbb{C}[z]$  (resp.  $\mathbb{C}[z^{-1}, z]$ ) the algebra of polynomials (resp. Laurent polynomials) in  $z$  over  $\mathbb{C}$ . For any  $1 \leq i \leq n$ , set  $\partial_i := \partial/\partial z_i$ , and  $\partial := (\partial_1, \partial_2, \dots, \partial_n)$ .

We say a differential operator  $\Lambda$  of  $\mathbb{C}[z]$  is a *differential operator with constant coefficients* if  $\Lambda = h(\partial)$  for some polynomial  $h(\xi) \in \mathbb{C}[\xi]$ , where  $\xi = (\xi_1, \xi_2, \dots, \xi_n)$  are another  $n$  commutative variables which also commute with  $z$ . For convenience, we will denote the polynomial  $h(\partial)$  by  $\Lambda(\partial)$  and simply write  $\Lambda = \Lambda(\partial)$ .

In this paper, we will prove four cases for the following *vanishing conjecture* of differential operators with constant coefficients proposed by the third author in [Zh3].

**Conjecture 1.1.** *Let  $P(z) \in \mathbb{C}[z]$  and  $\Lambda = \Lambda(\partial)$  for some  $\Lambda(\xi) \in \mathbb{C}[\xi]$ . Assume that  $\Lambda^m(P^m) = 0$  for any  $m \geq 1$ . Then  $\Lambda^m(P^{m+1}) = 0$  when  $m \gg 0$ .*

Actually, all the cases of the conjecture above that we will prove in this paper also hold in the following more general form.

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2000 *Mathematics Subject Classification.* 33C45, 32W99, 14R15.

*Key words and phrases.* The vanishing conjecture of differential operators with constant coefficients, polytopes of Laurent polynomials, the Jacobian conjecture.

The second author is supported by the Netherlands Organization of Scientific Research (NWO). The third author has been partially supported by NSA Grant R1-07-0053.

**Conjecture 1.2.** *Let  $P(z) \in \mathbb{C}[z]$  and  $\Lambda = \Lambda(\partial)$  for some  $\Lambda(\xi) \in \mathbb{C}[\xi]$ . Assume that  $\Lambda^m(P^m) = 0$  for any  $m \geq 1$ . Then, for any  $g(z) \in \mathbb{C}[z]$ , we have  $\Lambda^m(P^m g) = 0$  when  $m \gg 0$ .*

Note that, Conjecture 1.1 is just the special case of Conjecture 1.2 with  $g(z) = P(z)$ . Also, when  $\Lambda$  is a homogeneous differential operator of order 2 with constant coefficients, these two conjectures are actually equivalent (See [EZ] and [Zh3]).

Note also that Conjecture 1.2 has recently been generalized by the third author [Zh4] to the so-called *image conjecture* of commuting differential operators of order one with constant leading coefficients. Actually, Conjecture 1.2 is equivalent to the image conjecture of the commuting differential operators  $\xi_i - \partial_i$  ( $1 \leq i \leq n$ ) of the polynomial algebra  $\mathbb{C}[\xi, z]$  for the separable polynomial  $\Lambda(\xi)P(z) \in \mathbb{C}[\xi, z]$ . For more details, see [Zh4].

The main motivation behind Conjecture 1.1 is its connection with the well-known *Jacobian conjecture* proposed by O. H. Keller [K] in 1939 (See also [BCW] and [E]). The connection is given by the following theorem proved in [Zh2].

**Theorem 1.3.** *Let  $\Delta_n := \sum_{i=1}^n \partial_i^2$  be the Laplace operator of the polynomial algebra  $\mathbb{C}[z]$ . Then, the Jacobian conjecture holds for all  $n \geq 1$  iff Conjecture 1.1 with  $\Lambda = \Delta_n$  holds for all  $n \geq 1$ .*

It has also been shown in [Zh3] that one may replace the Laplace operators  $\Delta_n$  in the theorem above by any sequence  $\Lambda_n = \Lambda_n(\partial)$  ( $n \geq 1$ ) of differential operators with  $\Lambda_n(\xi)$  homogeneous of degree 2 whose ranks go to  $\infty$  as  $n \rightarrow \infty$ .

The proof of Theorem 1.3 is based on the remarkable symmetric reduction achieved independently by M. de Bondt and the first author [BE1] and G. Meng [Me]. It also depends on some results obtained in [Zh1] on a deformation of polynomial maps.

Currently, there are only a few cases of Conjecture 1.1 that are known. The best results so far come from M. de Bondt and the first author's results [BE2] [BE3] on symmetric polynomial maps via the equivalence obtained by the third author in [Zh2] and [Zh3]. The results state that Conjecture 1.1 holds for homogeneous quadratic differential operators  $\Lambda = \Lambda(\partial)$  if either  $n \leq 4$  (for any  $P(z) \in \mathbb{C}[z]$ ) or  $n \leq 5$  with  $P(z)$  homogeneous. The case when  $\Lambda(\xi)$  is an (integral) power of a homogeneous linear polynomial of  $\xi$  is an easy exercise (See also [Zh3]).

In this paper, we will prove four more cases of Conjecture 1.2 and also a conjecture proposed in [Zh3] by the third author on the Laurent polynomials with no holomorphic parts (See Theorem 5.1).

First, in Section 2, we use a fundamental theorem, Theorem 2.1, in ODE to show that Conjecture 1.2 holds for one variable case (See Theorem 2.3). Actually, in this case Conjecture 1.2 even holds for all formal power series  $P(z)$  and polynomials  $g(z) \in \mathbb{C}[z]$ .

In Section 3, we assume  $n = 2$  and show in Theorem 3.1 that Conjecture 1.2 holds for any differential operator  $\Lambda$  of the form  $\partial_x - \Phi(\partial_y)$ , where, for convenience, in this section we use  $(x, y)$  instead of  $(z_1, z_2)$  to denote two free commutative variables and  $\Phi(\cdot)$  to denote any polynomial in one variable. We also give a counter-example, Example 3.5, to show that both Conjectures 1.1 and 1.2 fail for some formal power series  $P(x, y) \in \mathbb{C}[[x, y]]$ .

In Section 4, we first recall and prove some results on rational polytopes, polytopes with all its vertices having rational coordinates. We then use Duistermaat and van der Kallen's remarkable theorem [DK] (See Theorem 4.5) to show what we call the *density theorem* of polytopes of Laurent polynomials (See Theorem 4.6). We will also show in Lemma 4.9 that Conjecture 1.2 holds when the polytope  $\text{Poly}(P) - \text{Poly}(\Lambda)$  has no intersection points with  $(\mathbb{R}^{\geq 0})^{\times n}$ . At the end of this section, we give a counter-example, Example 4.11, to show that Duistermaat and W. van der Kallen's theorem also fails for some Laurent formal power series.

In Section 5, we first give a proof for a conjecture proposed in [Zh3] on Laurent polynomials with no holomorphic parts (See Theorem 5.1). We then show in Corollary 5.3 that Conjecture 1.2 holds when either  $\Lambda(\xi)$  or  $P(z)$  is a monomial of  $\xi$  or  $z$ , respectively.

In Section 6, we show in Theorem 6.1 and Corollary 6.5 that Conjecture 1.2 holds when either  $\Lambda(\xi)$  or  $P(z)$  is a linear combination of two monomials of  $\xi$  or  $z$ , respectively, with different degrees.

**Acknowledgment** The authors would like to thank Harm Derksen, Wilberd van der Kallen and Han Peters for personal communications.

## 2. Proof of the Vanishing Conjecture for One Variable Case

In this section, we consider Conjecture 1.2 for the one-variable case and show that it does hold even for formal power series  $P(z)$  (See Theorem 2.3).

Throughout this section we assume that  $n = 1$  and let  $z$  denote a single free variable. For convenience, we also set  $D := d/dz$ . A different operator  $\Lambda = \Lambda(D)$  is also fixed, where  $\Lambda(\xi)$  denotes any non-zero polynomial in one variable.

With the notation above, let us first recall the following two well-known fundamental results from ODE (See [L] or any standard text book on ODE).

**Theorem 2.1.** *Let  $\lambda_i$  ( $1 \leq i \leq k$ ) be the set of all distinct roots of the polynomial  $\Lambda(\xi) \neq 0$  with multiplicity  $m_i \geq 1$ . Then, a formal power series  $P(z) \in \mathbb{C}[[z]]$  satisfies the differential equation  $\Lambda(D)P(z) = 0$  iff  $P(z)$  can be written as a linear combination over  $\mathbb{C}$  of  $z^j e^{\lambda_i z}$  ( $1 \leq i \leq k$ ;  $0 \leq j \leq m_i - 1$ ).*

**Lemma 2.2.** *For any distinct  $\lambda_i \in \mathbb{C}$  ( $1 \leq i \leq k$ ), the formal power series  $\{e^{\lambda_i z} \mid 1 \leq i \leq k\}$  are linearly independent over the rational function field  $\mathbb{C}(z)$ .*

In case that a formal power series  $P(z)$  can be written (uniquely) as

$$(2.1) \quad P(z) = \sum_{i=1}^k c_i(z) e^{\lambda_i z}$$

for some distinct  $\lambda_i \in \mathbb{C}$  ( $1 \leq i \leq k$ ) and non-zero  $c_i(z) \in \mathbb{C}[z]$ , we call this expression the *exponential expansion* of  $P(z)$ .

The main result of this section is the following theorem.

**Theorem 2.3.** *For any formal power series  $P(z)$  with  $\Lambda^m(P^m) = 0$  for any  $m \geq 1$ , we have*

- (a)  *$P(z)$  must be a polynomial in  $z$ .*
- (b) *Conjecture 1.2 holds for  $\Lambda$ ,  $P(z)$  and any  $g(z) \in \mathbb{C}[z]$ .*

*Proof:* In the proof below, we will freely use the notation fixed above for the differential operator  $\Lambda = \Lambda(D)$  and the polynomial  $\Lambda(\xi)$ .

Note first that, if  $P(z) = 0$ , there is nothing to prove. So we assume  $P(z) \neq 0$ .

Since  $\Lambda P = 0$ , by Theorem 2.1 and without losing any generality, we may write  $P(z)$  uniquely as in Eq. (2.1) with  $\lambda_i$  ( $1 \leq i \leq k$ ) being distinct roots of  $\Lambda(\xi)$  and  $c_i(z) \in \mathbb{C}[z]$  with  $\deg c_i(z) \leq m_i - 1$  for any  $1 \leq i \leq k$ .

Assume that  $P(z)$  is not a polynomial. Then, by Lemma 2.2, there exists  $\lambda_i \neq 0$  for some  $1 \leq i \leq k$ . Identify  $\mathbb{C}$  with  $\mathbb{R}^2$  and let  $\Sigma$  be the polytope or the convex subset in  $\mathbb{R}^2$  generated by  $\lambda_i$  ( $1 \leq i \leq k$ ). Then, there exists a vertex of  $\Sigma$  which is not the origin  $0 \in \mathbb{R}^2$ . Without losing any generality, we assume that  $\lambda_1$  is such a vertex.

For any  $m \geq 1$ , from Eq. (2.1), it is easy to see that  $P^m(z)$  also has an exponential expansion in which  $e^{m\lambda_1 z}$  appears with the nonzero coefficient  $c_{\lambda_1}^m(z)$ .

On the other hand, since  $\Lambda^m(P^m) = 0$ , by Theorem 2.1 and Lemma 2.2, we know that  $m\lambda_1$  must be a root of the polynomial  $\Lambda^m(\xi)$ , hence also a root of  $\Lambda(\xi)$ . Since  $\lambda_1 \neq 0$  and the statement above holds for any  $m \geq 1$ , we see that  $\Lambda(\xi)$  has infinitely many distinct roots  $m\lambda_1$  ( $m \geq 1$ ), which is impossible. Therefore,  $P(z)$  must be a polynomial and (a) holds.

To show (b), by (a) and Lemma 2.2, it is easy to see that none of non-zero roots of  $\Lambda(\xi)$  can be involved in Eq. (2.1). Hence we have  $k = 1$ ,  $\lambda_1 = 0$  and  $P(z) = c_1(z)$  with the degree

$$(2.2) \quad d := \deg P(z) = \deg c_1(z) \leq m_1 - 1.$$

Since  $\Lambda(\xi)$  has the root  $\lambda_1 = 0$  with multiplicity  $m_1 \geq 1$ , we may write  $\Lambda(\xi)$  as  $\Lambda(\xi) = \Phi(\xi)\xi^{m_1}$  for some  $\Phi(\xi) \in \mathbb{C}[\xi]$ . Consequently, we have  $\Lambda = \Lambda(D) = \Phi(D)D^{m_1}$ .

Now we fix any  $g(z) \in \mathbb{C}[z]$  with  $d' := \deg g(z) \geq 0$ . Then, for any  $m \geq 1$ , the polynomial  $D^{mm_1}(P^m g(z))$ , if not zero, has the degree

$$(2.3) \quad \deg D^{mm_1}(P^m g(z)) = (md + d') - mm_1 = d' - (m_1 - d)m.$$

Note that, by Eq. (2.2), we know that  $m_1 - d \geq 1$ . So, for any  $m > d'/(m_1 - d)$ , we have  $d' - (m_1 - d)m < 0$ . Furthermore, by Eq. (2.3), we have, for any  $m > d'/(m_1 - d)$ ,  $D^{mm_1}(P^m g(z)) = 0$ , hence also

$$\Lambda^m(P^m g(z)) = \Phi^m D^{mm_1}(P^m g(z)) = 0.$$

Therefore, Conjecture 1.2 holds for any  $g(z) \in \mathbb{C}[z]$ .  $\square$

Two remarks about Theorem 2.3 are as follows.

First, the theorem does not always hold for formal power series  $g(z) \in \mathbb{C}[[z]]$ . For example, let  $c \in \mathbb{C}$  such that  $c$  is not a root of  $\Lambda(\xi)$  and  $g(z) = e^{cz}$ . Then, By Theorem 2.3, (a) and Theorem 2.1, it is easy to see that  $\Lambda^m(P^m(z)g(z)) \neq 0$  for any  $m \geq 1$ .

Second, even though the conjecture 1.2 fails for some formal power series  $g(z) \in \mathbb{C}[[z]]$ , by Theorem 2.3, (a), it still holds when  $g(z) = P(z)$  without assuming in advance that  $g(z) = P(z)$  is a polynomial. In other words, Conjecture 1.1 actually still holds for all formal power series  $P(z)$ .

### 3. Proof of the Vanishing Conjecture for the Differential Operator $\Lambda = \partial_x - \Phi(\partial_y)$

Throughout this section, we denote by  $(x, y)$  instead of  $(z_1, z_2)$  two commutative free variables and by  $\xi$  another free variable which commutes with  $x$  and  $y$ . Once and for all, we also fix an arbitrary polynomial  $\Phi(\xi)$  and write it as

$$(3.1) \quad \Phi(\xi) = q_0 + q_1\xi + \cdots + q_k\xi^k$$

for some  $k \geq 0$  and  $q_i \in \mathbb{C}$  ( $0 \leq i \leq k$ ). We will denote by  $o(\Phi(\xi))$  or  $o(\Phi)$  the *order* of the polynomial  $\Phi(\xi)$ .

In this section, we first give a proof of Conjecture 1.2 for the differential operator  $\Lambda = \partial_x - \Phi(\partial_y)$  (See the theorem below). We then give a counter-example (See Example 3.5) to show that both Conjectures 1.1 and 1.2 fail for some formal power series  $P(x, y) \in \mathbb{C}[[x, y]]$ .

The main result of this section is the following theorem.

**Theorem 3.1.** *The Conjecture 1.2 holds for the differential operator  $\Lambda = \partial_x - \Phi(\partial_y)$  and all polynomials  $P(x, y) \in \mathbb{C}[x, y]$ .*

Note first that, if  $P(x, y) = 0$ , there is nothing to prove for the theorem. So, for the rest of this section, we fix an arbitrary polynomial  $P(x, y) \in \mathbb{C}[x, y]$  and assume  $P(x, y) \neq 0$ .

In order to prove Theorem 3.1, we first need the following three lemmas.

**Lemma 3.2.** *Let  $\Lambda$  and  $0 \neq P(x, y) \in \mathbb{C}[x, y]$  as fixed above. Assume that  $\Lambda P = 0$ . Then,  $q_0 = 0$ , or equivalently, the order  $o(\Phi(\xi)) \geq 1$  if  $\Phi(\xi) \neq 0$ .*

*Proof:* Assume  $q_0 \neq 0$ . By Eq. (3.1), we have  $\Lambda = \partial_x - \Phi(\partial_y) = \partial_x - (q_0 + q_1\partial_y + \cdots + q_k\partial_y^k)$ . Let  $d = \deg P(x, y)$  and  $P_d(x, y)$  the homogeneous part of  $P(x, y)$  of degree  $d$ . Note that the highest degree part of  $\Lambda P$  is  $q_0 P_d(x, y)$  which is equal to zero since  $\Lambda P = 0$ . Since we have assumed  $q_0 \neq 0$  and  $P(x, y) \neq 0$ , we have  $P_d(x, y) = 0$ , which is a contradiction.  $\square$

**Lemma 3.3.** *Let  $\Lambda$  and  $0 \neq P(x, y) \in \mathbb{C}[x, y]$  as above with  $\Lambda P = 0$ . Then, we have  $P(x, y) = e^{x\Phi(\partial_y)}(f(y))$  for some  $f(y) \in \mathbb{C}[y]$ .*

*Proof:* Note first that,

$$\partial_x(e^{-x\Phi(\partial_y)}P) = e^{-x\Phi(\partial_y)}(\partial_x - \Phi(\partial_y))P = e^{-x\Phi(\partial_y)}(\Lambda P) = 0.$$

So  $e^{-x\Phi(\partial_y)}P$  is independent on  $x$ . Hence  $e^{-x\Phi(\partial_y)}P = f(y)$  for some  $f(y) \in \mathbb{C}[y]$ . Now, applying  $e^{x\Phi(\partial_y)}$  to both sides of the latter equation, we get  $P(x, y) = e^{x\Phi(\partial_y)}(e^{-x\Phi(\partial_y)}P) = e^{x\Phi(\partial_y)}(f(y))$ .  $\square$

Note that  $e^{x\Phi(\partial_y)}(f(y))$  is still a polynomial, because from Lemma 3.2 it follows that  $\Phi = 0$  or  $o(\Phi) \geq 1$ . So, in the first case  $P(x, y) = f(y)$  and in the latter there exists an  $m$  such that  $\Phi^m(\partial_y)(f(y)) = 0$ .

**Lemma 3.4.** *Let  $\Lambda$  and  $P(x, y)$  as in Lemma 3.3. Further assume  $o(\Phi) \geq 2$  and  $\Lambda P = \Lambda^2(P^2) = 0$ . Then, we have  $o(\Phi) > \deg f$  and  $P(x, y) = f(y)$ .*

*Proof:* First, we view  $\Lambda^2(P^2) = (\partial_x - \Phi(\partial_y))^2((e^{x\Phi(\partial_y)}f(y))^2)$  as a polynomial in  $\mathbb{C}[y][x]$ , and look at its constant term, which is

$$\begin{aligned} \Lambda^2(P^2)|_{x=0} &= 2f(y)\Phi^2(\partial_y)f(y) + 2(\Phi(\partial_y)f(y))^2 \\ &\quad - 4\Phi(\partial_y)(f(y)\Phi(\partial_y)f(y)) + \Phi^2(\partial_y)(f^2(y)) \\ &= 0 \end{aligned}$$

Let  $d = \deg f(y)$  and  $r = o(\Phi(\xi)) \geq 2$ , and assume that  $d \geq r$ . Write  $f(y) = c_0 + c_1y + \cdots + c_dy^d$  and  $\Phi(\partial_y) = q_r\partial_y^r + q_{r+1}\partial_y^{r+1} + \cdots + q_k\partial_y^k$  for some  $k \geq r$ .

Now, by looking at the leading coefficient of  $\Lambda^2(P^2)|_{x=0}$ , we get

$$\begin{aligned} 0 &= 2c_d^2q_r^2v \frac{d!}{(d-r)!} + 2c_d^2q_r^2 \left( \frac{d!}{(d-r)!} \right)^2 \\ &\quad - 4c_d^2q_r^2 \frac{d!}{(d-r)!} \frac{(2d-r)!}{(2d-2r)!} + c_d^2q_r^2 \frac{(2d)!}{(2d-2r)!}, \end{aligned}$$

where  $v = 0$  if  $d < 2r$  and  $v = \frac{(d-r)!}{(d-2r)!}$  if  $d \geq 2r$ .

Then, by the assumptions  $c_d \neq 0$  and  $q_r \neq 0$ , we further have

(3.2)

$$0 = \frac{2vd!}{(d-r)!} + 2 \left( \frac{d!}{(d-r)!} \right)^2 - \frac{4d!(2d-r)!}{(d-r)!(2d-2r)!} + \frac{(2d)!}{(2d-2r)!}.$$

But, on the other hand, for any  $r \geq 2$ , we also have

$$(3.3) \quad \frac{2vd!}{(d-r)!} + 2 \left( \frac{d!}{(d-r)!} \right)^2 > 0$$

$$(3.4) \quad - \frac{4d!(2d-r)!}{(d-r)!(2d-2r)!} + \frac{(2d)!}{(2d-2r)!} \geq 0.$$

The first inequality is obvious. The second inequality holds is because it is equivalent to  $\binom{2d}{r} \geq 4\binom{d}{r}$  which follows from the general identity

$\binom{2d}{r} \geq 2^r \binom{d}{r}$  for any  $r \geq 0$  and the assumption that  $r \geq 2$ . Note that the last inequality can be easily verified by the facts that  $2d-i \geq 2(d-i)$  for any  $0 \leq i \leq r-1$ .

From the inequalities in Eqs. (3.3) and (3.4), we see that Eq. (3.2) can not hold, which means that our assumption  $d \geq r$  can not hold. Therefore, we have  $o(\Phi) > \deg f(y)$ . Then  $\Phi(\partial_y)f(y) = 0$  and, by Lemma 3.3, we also have  $P(x, y) = f(y)$ .  $\square$

Now we are ready to prove Theorem 3.1.

*Proof of Theorem 3.1:* Let  $\Lambda = \partial_x - \Phi(\partial_y)$  and  $0 \neq P(x, y) \in \mathbb{C}[x, y]$  such that  $\Lambda^m(P^m) = 0$  for any  $m \geq 1$ . Fix any  $g(x, y) \in \mathbb{C}[x, y]$ , we want to show that  $\Lambda^m(P^m g) = 0$  when  $m \gg 0$ .

First, by Lemma 3.3, we know  $P = e^{x\Phi(\partial_y)}(f(y))$  for some  $f(y) \in \mathbb{C}[y]$ . Furthermore, from Lemma 3.2 it follows that either  $\Phi(\xi) = 0$  or  $o(\Phi(\xi)) \geq 1$ . If  $\Phi = 0$ , then  $\Lambda = \partial_x$  and  $P = f(y)$ . In this case it immediately follows that  $\Lambda^m(P^m g) = 0$  for any  $m \geq \deg_x g$ .

So from now on assume that  $\Phi \neq 0$  and  $o(\Phi) \geq 1$ . Let  $k = \deg \Phi$ . By Eq. (3.1), we have  $\Phi(\partial_y) = q_1 \partial_y + \cdots + q_k \partial_y^k$ .

If  $q_1 \neq 0$ , then we can perform the coordinate change  $(x, y) \rightarrow (x', y') = (x, y + q_1 x)$  to get  $\tilde{\Lambda} = \partial_{x'} - (q_2 \partial_{y'}^2 + \cdots + q_k \partial_{y'}^k)$ ,  $P(x, y) = P(x', y' - q_1 x') = \tilde{P}(x', y')$  and  $g(x, y) = g(x', y' - q_1 x') = \tilde{g}(x', y')$ . Let  $\tilde{\Phi}(\partial_{y'}) = q_2 \partial_{y'}^2 + \cdots + q_k \partial_{y'}^k$ , then  $\tilde{\Lambda} = \partial_{x'} - \tilde{\Phi}(\partial_{y'})$ .

Because that,  $\tilde{\Lambda}^m(\tilde{P}^m) = 0 \Leftrightarrow \Lambda^m(P^m) = 0$  and  $\tilde{\Lambda}^m(\tilde{P}^m \tilde{g}) = 0 \Leftrightarrow \Lambda^m(P^m g) = 0$ , so we may also assume that  $o(\Phi) \geq 2$ . Then, by Lemma 3.4, we have  $o(\Phi) > \deg f$  and  $P(x, y) = f(y)$ .

Now, for any  $m \geq 1$ , we have

$$\begin{aligned} \Lambda^m(f^m(y)g(x, y)) &= (\partial_x - \Phi(\partial_y))^m(f^m(y)g(x, y)) \\ &= \sum_{\substack{a+b=m \\ a, b \geq 0}} \partial_x^a \Phi^b(\partial_y) (f^m(y)g(x, y)) \\ &= \sum_{\substack{a+b=m \\ a, b \geq 0}} \Phi^b(\partial_y) (f^m(y) \partial_x^a g(x, y)). \end{aligned}$$

Note that, the general term  $\Phi^b(\partial_y) (f^m(y) \partial_x^a g(x, y))$  in the summation above is not equal to zero only if

$$(3.5) \quad \begin{cases} a \leq \deg_x g, \\ b o(\Phi) = (m - a) o(\Phi) \leq m \deg f + \deg_y g. \end{cases}$$



Assume the inequalities above hold, Then, we have

$$(3.6) \quad m(o(\Phi) - \deg f) \leq a o(\Phi) + \deg_y g \leq o(\Phi) \deg_x g + \deg_y g.$$

Since  $o(\Phi) > \deg f$  (as pointed out above), the combined inequality above is equivalent to

$$(3.7) \quad m \leq \frac{o(\Phi) \deg_x g + \deg_y g}{o(\Phi) - \deg f}.$$

Therefore, from the arguments above, we have  $\Lambda^m(f^m g) = 0$  for any

$$(3.8) \quad m > \frac{o(\Phi) \deg_x g + \deg_y g}{o(\Phi) - \deg f}.$$

Hence, we have proved Theorem 3.1.  $\square$

Next we give a counter-example to show that, for the two variable case, both Conjectures 1.1 and 1.2 fail when  $P(x, y)$  is allowed to be a formal power series, instead of just a polynomial.

**Example 3.5.** Let  $\Lambda = \partial_y \partial_x$  and  $P(x, y) = x + e^y$ . Then, for any  $m \geq 1$ , we have

$$\begin{aligned} \Lambda^m(P^m) &= \partial_y^m \partial_x^m (x + e^y)^m = \partial_y^m (m!) = 0, \\ \Lambda^m(P^{m+1}) &= \partial_y^m \partial_x^m (x + e^y)^{m+1} = (m+1)! \partial_y^m (x + e^y) \\ &= (m+1)! e^y \neq 0. \end{aligned}$$

Hence Conjecture 1.1 fails in this case.

Furthermore, let  $g(x, y) = x$ . Then, for any  $m \geq 1$ , we have

$$\begin{aligned} \Lambda^m(P^m g) &= \partial_y^m \partial_x^m (x(x + e^y)^m) = \partial_y^m \left( m!x + \binom{m}{1} \partial^{m-1} (x + e^y)^m \right) \\ &= m! \partial_y^m (x + m(x + e^y)) = m m! e^y \neq 0. \end{aligned}$$

Hence, Conjecture 1.2 also fails for the formal power series  $P(x, y) = x + e^y$ .

#### 4. Some Results on Rational Polytopes and the Density Theorem of Polytopes of Laurent Polynomials

In this section, we first recall and prove some results on rational polytopes of  $\mathbb{R}^n$  that will be needed later in this paper. We then use Duistermaat and W. van der Kallen's theorem, Theorem 4.5, to prove what we call the *density theorem* of polytopes of Laurent polynomials (See Theorem 4.6) along with some of its variations. We will also show in Lemma 4.9 that Conjecture 1.2 holds when the polytope  $\text{Poly}(P) - \text{Poly}(\Lambda)$  has no intersection points with  $(\mathbb{R}^{\geq 0})^{\times n}$ . Finally, we give a

counter-example, Example 4.11, to show that Duistermaat and W. van der Kallen's theorem does not hold for some Laurent formal power series.

First, let us fix the following notations and conventions which together with the notations fixed in Section 1 will be used throughout the rest of this paper.

### Notation and Convention:

- (1) We use  $x = (x_1, x_2, \dots, x_n)$  to denote the coordinates of the Euclidean space  $\mathbb{R}^n$ . For any  $u = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ , against the traditional notation we set  $|u| := \sum_{i=1}^n a_i$ . Note that  $|u|$  could be negative for some  $u \in \mathbb{R}^n$ .
- (2) For any non-zero  $u \in \mathbb{R}^n$ , we denote by  $R_u$  the ray with the  $0 \in \mathbb{R}^n$  as its (only) end point and passing through  $u$ . When  $u = 0$ , we let  $R_u$  denote the single point  $0 \in \mathbb{R}^n$ .
- (3) For any  $m \in \mathbb{Z}$ ,  $\beta \in \mathbb{R}^n$  and a subset  $A \subset \mathbb{R}^n$ , we set

$$mA := \{mu \mid u \in A\},$$

$$\beta \pm A := \{\beta \pm u \mid u \in A\}.$$

- (4) We denote by  $\mathbb{R}_{\geq 0}^n$  the set of the vectors in  $\mathbb{R}^n$  whose components are all non-negative. Furthermore, we introduce a partial order  $>$  for vectors in  $\mathbb{R}^n$  by setting, for any  $u, v \in \mathbb{R}^n$ ,  $u > v$  if  $u \neq v$  and  $u - v \in \mathbb{R}_{\geq 0}^n$ ; and  $u \geq v$  if  $u > v$  or  $u = v$ .
- (5) For any finite subset  $A = \{u_i \mid 1 \leq i \leq k\} \subset \mathbb{R}^n$ , we set

$$(4.1) \quad \text{Poly}(A) := \left\{ \sum_{i=1}^k c_i u_i \mid c_i \geq 0; \sum_{i=1}^k c_i = 1 \right\}.$$

We call the subset above the *polytope* generated by the points  $u_i \in \mathbb{R}^n$  ( $1 \leq i \leq k$ ), or simply, by the subset  $A$ . Throughout the paper, by a *polytope* we always mean a subset of  $\mathbb{R}^n$  of the form as in Eq. (4.1).

- (6) For any  $u \in \mathbb{R}^n$ , we say  $u$  is a *rational point* of  $\mathbb{R}^n$  if its all components are rational numbers. We say a polytope is *rational* if all its vertices are rational.
- (7) For any fixed Laurent polynomial  $P(z) \in \mathbb{C}[z^{-1}, z]$ , and any  $\alpha \in \mathbb{Z}^n$ , we denote by  $[z^\alpha]P(z)$  the coefficient of  $z^\alpha$  in  $P(z)$ . We define the *support* of  $P(z)$ , denoted by  $\text{Supp}(P)$ , to be the subset of  $\alpha \in \mathbb{Z}^n$  such that  $[z^\alpha]P(z) \neq 0$ ; and the *polytope* of  $P(z)$ , denote by  $\text{Poly}(P)$ , to be the polytope generated by  $\text{Supp}(P)$ .

- (8) For any differential operator  $\Lambda = \Lambda(\partial)$  with  $\Lambda(\xi) \in \mathbb{C}[\xi]$ , we define the *support* of  $\Lambda$ , denoted by  $\text{Supp}(\Lambda)$ , and the *polytope* of  $\Lambda$ , denoted by  $\text{Poly}(\Lambda)$ , to be the support and the polytope, respectively, of the polynomial  $\Lambda(\xi)$ .

We start with the following lemma which is well-known (e.g. see [CLO]) and also easy to prove directly. In our later argument we will frequently use this lemma without explicitly referring to it.

**Lemma 4.1.** (a) *For any polytope  $\Sigma$  in  $\mathbb{R}^n$  and  $m \geq 1$ , we have*

$$(4.2) \quad m\Sigma = \left\{ \sum_{i=1}^m u_i \mid u_i \in \Sigma \right\}.$$

- (b) *For any Laurent polynomial  $P(z) \in \mathbb{C}[z^{-1}, z]$  and  $m \geq 1$ , we have*

$$(4.3) \quad \text{Poly}(P^m) = m\text{Poly}(P).$$

The following lemma is also well-known. But, for the sake of completeness, we include a proof here.

**Lemma 4.2.** *For any  $\beta \in \mathbb{R}^n$  and any polytope  $\Sigma$  in  $\mathbb{R}^n$  with  $\Sigma \cap \mathbb{R}_{\geq 0}^n = \emptyset$ , there exists  $N \geq 1$  such that  $(\beta + m\Sigma) \cap \mathbb{R}_{\geq 0}^n = \emptyset$  for any  $m \geq N$ .*

*Proof:* Assume otherwise, then there exist a strictly increasing sequence  $\{m_k \mid m_k \in \mathbb{N}\}$  and a sequence  $\{v_k \mid v_k \in (\beta + m_k\Sigma) \cap \mathbb{R}_{\geq 0}^n\}$ .

Note that, for any  $k \geq 1$ , we may write  $v_k = \beta + m_k u_k$  for some  $u_k \in \Sigma$ . Since  $\Sigma$  is bounded and closed, and hence compact, the sequence  $\{u_k \mid k \geq 1\}$  has a subsequence which converges to an element  $u \in \Sigma$ . Without losing any generality, we still denote this subsequence by  $\{u_k \mid k \geq 1\}$ .

Note that, for any  $k \geq 1$ ,  $u_k = v_k/m_k - \beta/m_k$ . Then we have

$$u = \lim_{k \rightarrow \infty} u_k = \lim_{k \rightarrow \infty} (v_k/m_k - \beta/m_k) = \lim_{k \rightarrow \infty} (v_k/m_k).$$

Since, for any  $k \geq 1$ ,  $m_k \geq 1$  and  $v_k \in \mathbb{R}_{\geq 0}^n$ , we have  $v_k/m_k \in \mathbb{R}_{\geq 0}^n$ . Furthermore, since  $\mathbb{R}_{\geq 0}^n$  is closed, from the equation above we have  $u \in \mathbb{R}_{\geq 0}^n$ . Therefore, we have  $u \in \Sigma \cap \mathbb{R}_{\geq 0}^n$ , which is a contradiction. Hence the lemma holds.  $\square$

**Lemma 4.3.** *For any two rational polytopes  $\Sigma$  and  $\Gamma$ , we have*

- (a) *If  $\Sigma \cap \Gamma \neq \emptyset$ , then it is also a rational polytope.*
- (b)  *$\Sigma - \Gamma := \{u - v \mid u \in \Sigma, v \in \Gamma\}$  is also a rational polytope.*
- (c) *For any rational  $w \in \Sigma - \Gamma$ , there exist  $u \in \Sigma$  and  $v \in \Gamma$  such that  $u, v$  are both rational and  $w = u - v$ .*

*Proof:* (a) It is well known (e.g. see Theorem 1.1, pp. 29 in [Zi]) that any polytope is a set of common solutions of a system of linear equations or inequalities. It is easy to see that a polytope is rational iff its determining linear equations or inequalities are defined over  $\mathbb{Q}$ , i.e. all the coefficients of unknowns including constant terms of the equations or inequalities are in  $\mathbb{Q}$ . Since  $\Sigma \cap \Gamma$  is determined by the union of determining linear equations or inequalities of  $\Sigma$  and  $\Gamma$ ,  $\Sigma \cap \Gamma$  is also a rational polytope if it is not empty.

(b) First, it is easy to see that  $(-1)\Gamma$  is a rational polytope and  $\Sigma - \Gamma$  is the same as the so-called Minkowski sum of the polytopes  $\Sigma$  and  $\Gamma$ . It is well known that the Minkowski sum of any two polytopes is also a polytope (e.g. see [CLO] and [Zi]). The reason that the polytope  $\Sigma - \Gamma$  is also rational is because any vertex of  $\Sigma - \Gamma$  is the difference of a vertex of  $\Sigma$  and a vertex of  $\Gamma$ .

(c) Note that, for the fixed  $w \in \Sigma - \Gamma$  in the lemma, the set of elements  $u \in \Sigma$  such that  $w = u - v$  for some  $v \in \Gamma$  is given by  $\Sigma \cap (w + \Gamma)$  which is non-empty by the existence of  $w$  itself.

Since both  $w$  and  $\Gamma$  are rational, so is  $w + \Gamma$ . By (a) we know that  $\Sigma \cap (w + \Gamma)$  is a (non-empty) rational polytope since  $\Sigma$  is also rational.

Let  $u$  be any rational point of  $\Sigma \cap (w + \Gamma)$ , say a vertex of this polytope, and write it as  $u = w + v$  for some  $v \in \Gamma$ . Since  $v = u - w$ ,  $v$  is also rational. Hence we get (c).  $\square$

**Corollary 4.4.** *For any rational polytope  $\Sigma$  of  $\mathbb{R}^n$  with  $\Sigma \cap \mathbb{R}_{\geq 0}^n \neq \emptyset$ , there exists a rational  $u \in \Sigma \cap \mathbb{R}_{\geq 0}^n$ .*

*Proof:* Since the rational polytope  $\Sigma$  is a closed and bounded subset of  $\mathbb{R}^n$ , so is  $\Sigma \cap \mathbb{R}_{\geq 0}^n$ . We may choose an  $N \in \mathbb{N}$  such that  $\Sigma \cap \mathbb{R}_{\geq 0}^n$  lies inside the polytope generated by  $0 \in \mathbb{R}^n$  and  $Ne_i$  ( $1 \leq i \leq n$ ), where  $e_i$ 's are the vectors in the standard basis of  $\mathbb{R}^n$ . Since the latter polytope is also rational, the corollary follows from Lemma 4.3, (a).  $\square$

Next we prove some results for the polytopes of Laurent polynomials. But, first let us recall the following remarkable theorem which was first conjectured by O. Mathieu [Ma] and later was proved by J. Duistermaat and W. van der Kallen [DK].

**Theorem 4.5. (Duistermaat and van der Kallen)** *For any  $f(z) \in \mathbb{C}[z^{-1}, z]$  such that the constant term of  $f^m(z)$  is equal to zero for any  $m \geq 1$ , we have  $0 \notin \text{Poly}(f)$ .*

The following result will play some crucial roles in later sections. We believe that it is also important in its own right, so we formulate

it as a theorem and call it the *density theorem* of polytopes of Laurent polynomials.

**Theorem 4.6. (The Density Theorem)** *For any  $P(z) \in \mathbb{C}[z^{-1}, z]$  and any rational point  $u \in \text{Poly}(P)$ , there exists  $m \geq 1$  such that  $R_u \cap \text{Supp}(P^m) \neq \emptyset$ .*

*Proof:* Assume otherwise, i.e.  $R_u \cap \text{Supp}(P^m) = \emptyset$  for any  $m \geq 1$ .

Since  $u$  is rational, there exists  $N \geq 1$  such that  $\beta := Nu \in \mathbb{Z}^n$ . Hence, we have  $R_u = R_\beta$ . Since  $u \in \text{Poly}(P)$  and  $\text{Poly}(P^N) = N\text{Poly}(P)$ , we have

$$(4.4) \quad \beta = Nu \in \text{Poly}(P^N).$$

Let  $f(z) := z^{-\beta} P^N(z) \in \mathbb{C}[z^{-1}, z]$ . Then, for any  $m \geq 1$ , the constant term of  $f^m$  is equal to zero. Otherwise, we would have  $m\beta \in \text{Supp}(P^{mN})$  and  $m\beta \in R_\beta \cap \text{Supp}(P^{mN}) = R_u \cap \text{Supp}(P^{mN})$ , which contradicts to our assumption on  $R_u$ .

Now apply Theorem 4.5 to  $f(z)$ , we have  $0 \notin \text{Poly}(f)$ . But, on the other hand, we have

$$\text{Poly}(f) = -\beta + \text{Poly}(P^N).$$

Therefore,  $\beta \notin \text{Poly}(P^N)$  which contradicts to Eq.(4.4). Hence, we have proved the theorem.  $\square$

From the proof above, we can actually get a stronger result.

**Corollary 4.7.** *For any  $P(z) \in \mathbb{C}[z^{-1}, z]$  and any rational point  $u \in \text{Poly}(P)$ , there exist infinitely many  $m_i \geq 1$  such that*

$$R_u \cap \text{Supp}(P^{m_i}) \neq \emptyset.$$

*Proof:* Assume otherwise, then there exists  $N \geq 1$  such that  $R_u \cap \text{Supp}(P^m) = \emptyset$  for any  $m \geq N$ .

But, applying Theorem 4.6 to the Laurent polynomial  $P^N(z)$  and the rational point  $Nu \in \text{Poly}(P^N)$ , we see that there exists  $m \geq 1$ , such that

$$\emptyset \neq R_{Nu} \cap \text{Supp}(P^{mN}) = R_u \cap \text{Supp}(P^{mN}).$$

Since  $mN \geq N$ , we get a contradiction.  $\square$

When  $P(z)$  is a homogeneous with respect to the generalized degree (counting  $\deg z_i^{-1} = -1$  for any  $1 \leq i \leq n$ ) of Laurent polynomials, we have the following more precise result.

**Corollary 4.8.** *For any homogeneous Laurent polynomial  $P(z)$  of degree  $d \neq 0$ , and any rational point  $u \in \text{Poly}(P)$ , there exist infinitely many  $m_i \geq 1$  such that  $m_i u \in \text{Supp}(P^{m_i})$ .*

*Proof:* First, since  $\deg P = d$ , we see that  $\text{Poly}(P)$  lies in the affine hyperplane  $H$  determined by the equation  $\sum_{i=1}^n x_i = d$ . Since  $u \in \text{Poly}(P) \subset H$ , we have  $|u| = d$ .

Second, by Corollary 4.7, we know that there exist infinitely many  $m_i \geq 1$  such that  $R_u \cap \text{Supp}(P^{m_i}) \neq \emptyset$ . We fix any such an  $m_i \geq 1$  and choose any  $\beta_i \in R_u \cap \text{Supp}(P^{m_i})$  (Actually, such  $\beta_i$  is unique as the intersection point of  $R_u$  with the affine hyperplane  $m_i H$ ). Write  $\beta_i = k_i u$  for some  $k_i \geq 0$ . Then we have  $|\beta_i| = k_i |u| = k_i d$ . Since  $\beta_i \in \text{Supp}(P^{m_i}) \subset m_i \text{Poly}(P) \subset m_i H$ , we have  $|\beta_i| = m_i d$ . Hence we have  $k_i = m_i$  and  $\beta_i = m_i u$ .  $\square$

Next we prove the following general result on Conjecture 1.2, which will be needed in Section 6.

**Lemma 4.9.** *Let  $P(z) \in \mathbb{C}[z]$  and  $\Lambda = \Lambda(\partial)$  any differential operator with constant coefficients. Assume*

$$(4.5) \quad (\text{Poly}(P) - \text{Poly}(\Lambda)) \cap \mathbb{R}_{\geq 0}^n = \emptyset.$$

*Then, we have*

- (a) *for any  $m \geq 1$ ,  $\Lambda^m(P^m) = 0$ .*
- (b) *for any  $g(z) \in \mathbb{C}[z]$ ,  $\Lambda^m(P^m g) = 0$  when  $m \gg 0$ .*

*Proof:* We first prove (b) as follows.

First, note that, by the linearity of  $\Lambda^m(P^m g)$  ( $m \geq 1$ ) on  $g(z) \in \mathbb{C}[z]$ , it is easy to see that we may assume  $g(z) = z^\gamma$  for some  $\gamma \in \mathbb{N}^n$ .

Second, let  $\Sigma := \text{Poly}(P) - \text{Poly}(\Lambda)$ . Then, we have that  $\Sigma \cap \mathbb{R}_{\geq 0}^n = \emptyset$  and also, by Lemma 4.3, (b),  $\Sigma$  is a polytope of  $\mathbb{R}^n$ .

Apply Lemma 4.2 to the polytope  $\Sigma$  and  $\gamma \in \mathbb{N}^n$ , we get a  $N \geq 1$  such that, for any  $m \geq N$ ,

$$(4.6) \quad (\gamma + m\Sigma) \cap \mathbb{R}_{\geq 0}^n = \emptyset.$$

On the other hand, for any  $m \geq 1$ , we have

$$(4.7) \quad m\Sigma = m\text{Poly}(P) - m\text{Poly}(\Lambda) = \text{Poly}(P^m) - \text{Poly}(\Lambda^m).$$

Therefore, by Eq.(4.6) and the equation above, we have

$$(4.8) \quad \mathbb{R}_{\geq 0}^n \cap (\gamma + (\text{Poly}(P^m) - \text{Poly}(\Lambda^m))) = \emptyset$$

for any  $m \geq N$ .

Consequently, for any  $m \geq N$ ,  $\lambda \in \text{Supp}(P^m) \subset \text{Poly}(P^m)$  and  $\mu \in \text{Supp}(\Lambda^m) \subset \text{Poly}(\Lambda^m)$ , we have

$$(4.9) \quad (\gamma + \lambda) - \mu = \gamma + (\lambda - \mu) \notin \mathbb{R}_{\geq 0}^n.$$

Hence, we have  $\partial^\mu z^{\gamma+\lambda} = 0$ . Since  $\Lambda^m$  is a linear combination of  $\partial^\mu$  ( $\mu \in \text{Supp}(\Lambda^m)$ ) and  $z^\gamma P^m(z)$  is a linear combination over  $\mathbb{C}$  of  $z^{\gamma+\lambda}$  ( $\lambda \in \text{Supp}(P^m)$ ), we have  $\Lambda^m(z^\gamma P^m(z)) = 0$  for any  $m \geq N$ . Therefore, we have proved (b).

To see (a) also holds, note that, by choosing  $\gamma = 0 \in \mathbb{N}^n$  in the proof above, Eq.(4.6) actually holds for any  $m \geq 1$ . This is because the condition  $\Sigma \cap \mathbb{R}_{\geq 0}^n = \emptyset$  implies directly  $m\Sigma \cap \mathbb{R}_{\geq 0}^n = \emptyset$  for any  $m \geq 1$ . Therefore, the argument above also goes through with  $N = 1$  and  $\gamma = 0$ , which means (a) also holds.  $\square$

**Remark 4.10.** *Note that, by Lemma 4.9, (a), we see that the condition given in Eq. (4.5) implies the condition in Conjecture 1.2. By Lemma 4.9, (b), we see that Conjecture 1.2 does hold in this case.*

Finally, let us point out that, like Conjectures 1.1 and 1.2, Duistermaat and W. van der Kallen's theorem, Theorem 4.5, also fails for some Laurent formal power series in two variables.

**Example 4.11.** *Let  $f(x, y) = y^{-1}(1 + x^{-1}e^y)$  and  $g(x, y) = x$ . Then, for any  $m \geq 1$ , it is easy to check that, the constant term of  $f^m$  is equal to zero, but the constant term of  $f^m g$  is equal to  $1/(m-1)! \neq 0$ . Therefore,  $0 \in \text{Supp}(f^m g)$  for any  $m \geq 1$ .*

*Assume that Theorem 4.5 holds for  $f(x, y)$ , i.e.  $0 \notin \text{Poly}(f)$ , then, as shown in [DK], we will have  $0 \notin \text{Supp}(f^m g)$  when  $m \gg 0$ . But this contradicts to the fact pointed out above, i.e.  $0 \in \text{Supp}(f^m g)$  for any  $m \geq 1$ . Hence Theorem 4.5 fails for  $f(x, y)$ .*

## 5. A Theorem on Laurent Polynomials with no Holomorphic Parts and the Vanishing Conjecture when $\Lambda(\partial)$ or $P(z)$ is a Monomial

In this section, we first prove a conjecture proposed in [Zh3] on Laurent polynomials with no holomorphic parts (See Theorem 5.1). By using this result, we then show in Corollary 5.3 that Conjecture 1.2 holds when the polynomials  $P(z)$  or the differential operators  $\Lambda = \Lambda(\partial)$  is a monomial of  $z$  or  $\partial$ , respectively.

**Theorem 5.1.** *Let  $f(z) \in \mathbb{C}[z^{-1}, z]$  such that, for any  $m \geq 1$ , the holomorphic part of  $f^m$  is equal to zero, i.e.  $\text{Supp}(f^m) \cap \mathbb{R}_{\geq 0}^n = \emptyset$ ,*

or equivalently,  $[z^\alpha]f^m = 0$  for any  $\alpha \in \mathbb{N}^n$ . Then, for any  $g(z) \in \mathbb{C}[z^{-1}, z]$ , the holomorphic part of  $gf^m$  is equal to zero when  $m \gg 0$ .

*Proof:* Since the subset of Laurent polynomials with no holomorphic parts is a subspace of  $\mathbb{C}[z^{-1}, z]$ , and  $g(z)$  is a linear combination over  $\mathbb{C}$  of finitely many monomials of  $z$ . It is easy to see that we may assume  $g(z) = z^\beta$  for some  $\beta \in \mathbb{Z}^n$ .

Let us first show  $\text{Poly}(f) \cap \mathbb{R}_{\geq 0}^n = \emptyset$  by the contradiction method.

Assume otherwise, i.e.  $\text{Poly}(f) \cap \mathbb{R}_{\geq 0}^n \neq \emptyset$ . Since  $\text{Poly}(f)$  is a rational polytope, by Corollary 4.4, there exists a rational  $u \in \text{Poly}(f) \cap \mathbb{R}_{\geq 0}^n$ . Below we fix any such a rational point  $u$ .

Apply Theorem 4.6 to  $f(z) \in \mathbb{C}[z^{-1}, z]$  and the rational point  $u \in \text{Poly}(f)$ , there exists  $N \geq 1$  such that  $R_u \cap \text{Supp}(f^N) \neq \emptyset$ . Note that  $R_u \subset \mathbb{R}_{\geq 0}^n$  since  $u \in \mathbb{R}_{\geq 0}^n$ . So we have  $\mathbb{R}_{\geq 0}^n \cap \text{Supp}(f^N) \neq \emptyset$ , and hence the holomorphic part of  $f^N$  is not zero, which is a contradiction. Therefore, we must have  $\text{Poly}(f) \cap \mathbb{R}_{\geq 0}^n = \emptyset$ .

Next, apply Lemma 4.2 to the polytope  $\text{Poly}(f)$  and  $\beta \in \mathbb{N}^n$ , we know that  $\mathbb{R}_{\geq 0}^n \cap (\beta + m\text{Poly}(f)) = \emptyset$  when  $m \gg 0$ . Since, for any  $m \geq 1$ ,  $m\text{Poly}(f) = \text{Poly}(f^m)$  and  $\text{Supp}(f^m) \subset \text{Poly}(f^m)$ , we have  $\mathbb{R}_{\geq 0}^n \cap (\beta + \text{Supp}(f^m)) = \emptyset$  when  $m \gg 0$ .

On the other hand, it is easy to see that  $\text{Supp}(z^\beta f^m) = \beta + \text{Supp}(f^m)$  for any  $m \geq 1$ . Therefore, we have  $\mathbb{R}_{\geq 0}^n \cap \text{Supp}(z^\beta f^m) = \emptyset$  when  $m \gg 0$ , which means that the theorem holds for  $g(z) = z^\beta$ .  $\square$

Next, we use Theorem 5.1 to show that Conjecture 1.2 holds when  $P(z)$  or  $\Lambda$  is a monomial of  $z$  or  $\partial$ , respectively. But, first, let us formulate the following simple observation as a lemma since it will be crucial for our later arguments.

**Lemma 5.2.** *For any  $\alpha, \beta \in \mathbb{N}^n$ , we have that,  $\partial^\alpha z^\beta = 0$  iff  $\beta \not\geq \alpha$ , or equivalently,  $\beta - \alpha \notin \mathbb{R}_{\geq 0}^n$ .*

Note that the lemma above is not necessarily true for  $\beta \in \mathbb{Z}^n \setminus \mathbb{N}^n$ .

**Corollary 5.3.** *Conjecture 1.2 holds if either  $\Lambda = \partial^\alpha$  or  $P(z) = z^\alpha$  for some  $\alpha \in \mathbb{N}^n$ .*

*Proof:* We prove the corollary for the case that  $P(z) = z^\alpha$  for some  $\alpha \in \mathbb{N}^n$ . The proof for the other case is similar.

First, by the linearity on  $g(z) \in \mathbb{C}[z]$ , we may assume that  $g(z) = z^\gamma$  for some  $\gamma \in \mathbb{N}^n$ .

Second, for any  $h(\xi) \in \mathbb{C}[\xi]$  and any  $\beta \in \mathbb{N}^n$ , by Lemma 5.2, it is easy to see that we have the following equivalences:

$$(5.1) \quad h(\partial)z^\beta = 0 \quad \Leftrightarrow \quad \beta \not\geq \mu \text{ for any } \mu \in \text{Supp}(h)$$



$$\Leftrightarrow \text{Supp}(z^\beta h(z^{-1})) \cap \mathbb{R}_{\geq 0}^n = \emptyset.$$

Note that the last statement above is equivalent to saying that the holomorphic part of the Laurent polynomial  $z^\beta h(z^{-1})$  is equal to zero.

Now, we write  $\Lambda = \Lambda(\partial)$  for some polynomial  $\Lambda(\xi) \in \mathbb{C}[\xi]$  and set  $f(z) := \Lambda(z^{-1})z^\alpha$ . Since  $\Lambda^m(P^m) = \Lambda^m(z^{m\alpha}) = 0$  for any  $m \geq 1$ , applying the equivalences in Eq. (5.1) with  $h(\xi) = \Lambda^m(\xi)$  and  $\beta = m\alpha$ , we know that the holomorphic part of  $f^m = \Lambda^m(z^{-1})z^{m\alpha}$  is equal to zero for any  $m \geq 1$ .

Applying Theorem 5.1 to  $f(z)$  and  $g(z) = z^\gamma$ , we know that there exists a  $N \geq 1$  such that the holomorphic part of  $f^m z^\gamma = \Lambda^m(z^{-1})z^{m\alpha+\gamma}$  is equal to zero for any  $m \geq N$ . For any fixed  $m \geq N$ , applying the equivalences in Eq. (5.1) with  $h(\xi) = \Lambda^m(\xi)$  and  $\beta = m\alpha + \gamma$ , we get  $\Lambda^m(P^m z^\gamma) = \Lambda^m(z^{m\alpha+\gamma}) = 0$ . Therefore, Conjecture 1.2 does hold in this case.  $\square$

## 6. Proof of the Vanishing Conjecture for the Differential Operator $\Lambda = a\partial^\alpha + b\partial^\beta$ with $|\alpha| \neq |\beta|$

In this section, we prove the following case of Conjecture 1.2.

**Theorem 6.1.** *Let  $\alpha, \beta \in \mathbb{N}^n$  such that  $|\alpha| \neq |\beta|$ , and  $\Lambda = a\partial^\alpha + b\partial^\beta$  for some  $a, b \in \mathbb{C}$ . Then, Conjecture 1.2 holds for the differential operator  $\Lambda$  and any homogeneous  $P(z) \in \mathbb{C}[z]$ .*

First, let us consider the following simple cases of the theorem above.

If  $a = b = 0$ , then  $\Lambda = 0$ . There is nothing to prove. If only one of the  $a$  and  $b$  is zero, then, after a change of variables, we may assume  $\Lambda = \partial^\gamma$  for some  $\gamma \in \mathbb{N}^n$ . Then Conjecture 1.2 in this case follows directly from Corollary 5.3.

Therefore we may assume that  $a$  and  $b$  are both nonzero. Then, by using the fact that  $\alpha \neq \beta$ , it is easy to see that, after a change of variables, we may assume  $a = b \neq 0$ . Because of that, Conjecture 1.2 holds for any differential operator iff it holds for any nonzero scalar multiple of the differential operator, we may further assume that  $a = b = 1$ . Therefore, Theorem 6.1 is reduced to the case when  $\Lambda = \partial^\alpha + \partial^\beta$  for some  $\alpha, \beta \in \mathbb{N}^n$  with  $|\alpha| \neq |\beta|$ .

Throughout the rest of this section, we will fix a differential operator  $\Lambda$  as above and a homogeneous polynomial  $P(z) \in \mathbb{C}[z]$  of degree  $d = \deg P \geq 0$  such that  $\Lambda^m(P^m) = 0$  for any  $m \geq 1$ . We divide the proof of Theorem 6.1 for this case into several lemmas.

**Lemma 6.2.** (a) *For any  $m \geq 1$ , we have*

$$(6.1) \quad \text{Supp}(\Lambda^m) = \{k\alpha + \ell\beta \mid k, \ell \in \mathbb{N}; k + \ell = m\}.$$

(b) For any  $m, k \geq 1$  and  $u \in \text{Supp}(\Lambda^k)$ ,  $mu \in \text{Supp}(\Lambda^{mk})$ .

*Proof:* (a) The statement follows directly from the binomial expansion of  $\Lambda^m = (\partial^\alpha + \partial^\beta)^m$ .

(b) By (a), we may write  $u = r\alpha + s\beta$  for some  $r, s \in \mathbb{N}$  with  $r + s = k$ . Then, we have  $mu = mr\alpha + ms\beta$ . Since  $mr, ms \in \mathbb{N}$  and  $mr + ms = mk$ , by (a) again we have  $mu \in \text{Supp}(\Lambda^{mk})$ .  $\square$

**Lemma 6.3.** For any fixed  $m \geq 1$ , we have

(a)  $\partial^\mu P^m = 0$  for any  $\mu \in \text{Supp}(\Lambda^m)$ .

(b) for any  $\mu \in \text{Supp}(\Lambda^m)$  and  $\gamma \in \text{Supp}(P^m)$ , we have  $\gamma \not\geq \mu$  or equivalently,  $\gamma - \mu \notin \mathbb{R}_{\geq 0}^n$ .

*Proof:* (a) We first consider

$$(6.2) \quad 0 = \Lambda^m(P^m) = (\partial^\alpha + \partial^\beta)^m(P^m) = \sum_{\substack{k, \ell \geq 0 \\ k + \ell = m}} \binom{m}{k} \partial^{k\alpha + \ell\beta}(P^m).$$

Note that, for any  $k, \ell \geq 0$  with  $k + \ell = m$ , we have

$$(6.3) \quad \deg(\partial^{k\alpha + \ell\beta} P^m) = md - (k|\alpha| + \ell|\beta|).$$

Assume that, for some  $k', \ell' \in \mathbb{N}^n$  with  $k' + \ell' = m$ , we have

$$\deg(\partial^{k'\alpha + \ell'\beta} P^m) = \deg(\partial^{k\alpha + \ell\beta} P^m)$$

Then, by Eq. (6.3), we get

$$(6.4) \quad \begin{aligned} k|\alpha| + \ell|\beta| &= k'|\alpha| + \ell'|\beta|, \\ (k - k')|\alpha| &= (\ell' - \ell)|\beta|. \end{aligned}$$

Since  $k + \ell = k' + \ell' = m$ , we have  $k - k' = \ell' - \ell$ . Combining Eq. (6.4) with the fact that  $|\alpha| \neq |\beta|$ , we see that  $k - k' = \ell' - \ell = 0$ . Therefore, all the terms in the sum of Eq. (6.2) have different degrees. Hence they all have to be zero.

On the other hand, by Lemma 6.2, (a), any  $\mu \in \text{Supp}(\Lambda^m)$  has the form  $k\alpha + \ell\beta$  for some  $k, \ell \in \mathbb{N}$  with  $k + \ell = m$ . Hence (a) follows.

(b) First, we write  $P^m(z)$  as  $P^m(z) = \sum_{\gamma \in \text{Supp}(P^m)} b_\gamma z^\gamma$  with  $b_\gamma \in \mathbb{C}^\times$ . For any  $\mu \in \text{Supp}(\Lambda^m)$ , by (a) and also Lemma 5.2, we have

$$(6.5) \quad 0 = \partial^\mu(P^m) = \sum_{\gamma \in \text{Supp}(P^m)} b_\gamma \partial^\mu(z^\gamma) = \sum_{\substack{\gamma \in \text{Supp}(P^m) \\ \gamma \geq \mu}} b_\gamma z^{\gamma - \mu}.$$

Note that  $b_\gamma \neq 0$  for any  $\gamma \in \text{Supp}(P^m)$  and, for any  $\gamma_1 \neq \gamma_2 \in \text{Supp}(P^m)$ ,  $z^{\gamma_1 - \mu} \neq z^{\gamma_2 - \mu}$ . Then, from Eq. (6.5) we see that there can not be any  $\gamma \in \text{Supp}(P^m)$  such that  $\gamma \geq \mu$ . Hence (b) also holds.  $\square$

**Lemma 6.4.** *Let  $u \in \text{Poly}(P)$  and  $v \in \text{Poly}(\Lambda)$ . Assume that both  $u$  and  $v$  are rational. Then, we have,  $u \not\geq v$ .*

*Proof:* First, denote by  $L_{\alpha,\beta}$  the line segment in  $\mathbb{R}^n$  connecting  $\alpha$  and  $\beta$ . Then it is easy to see that  $\text{Poly}(\Lambda) = L_{\alpha,\beta}$ . Since both  $\alpha$  and  $\beta$  are rational, it is easy to check that, a point of  $L_{\alpha,\beta}$  is rational iff it is a linear combination of  $\alpha$  and  $\beta$  with rational coefficients. Therefore, we may write  $v = r\alpha + s\beta$  for some rational  $r, s \geq 0$  with  $r + s = 1$ . Let  $N \geq 1$  such that  $Nr, Ns \in \mathbb{N}$ . Since  $Nr + Ns = N$ , by Lemma 6.2, (a), we have  $Nv = Nr\alpha + Ns\beta \in \text{Supp}(\Lambda^N)$ .

Second, apply Corollary 4.8 to the homogeneous polynomial  $P^N$  and the rational point  $Nu \in \text{Poly}(P^N)$ , there exists  $m \geq 1$  such that  $mNu \in \text{Supp}(P^{mN})$ . Since  $Nv \in \text{Supp}(\Lambda^N)$ , by Lemma 6.2, (b), we have  $mNv \in \text{Supp}(\Lambda^{mN})$ .

Note that  $mNu \in \text{Supp}(P^{mN})$  and  $mNv \in \text{Supp}(\Lambda^{mN})$  are both rational, by Lemma 6.3, (b), we have  $mNu - mNv = mN(u - v) \notin \mathbb{R}_{\geq 0}^n$ . Since  $mN > 0$ , we also have  $(u - v) \notin \mathbb{R}_{\geq 0}^n$ , i.e.  $u \not\geq v$ .  $\square$

Now, we can prove the main result, Theorem 6.1, of this section as follows.

*Proof of Theorem 6.1:* First, by the reductions given at the beginning of this section, we may assume that the differential operator  $\Lambda = \partial^\alpha + \partial^\beta$  for some  $\alpha, \beta \in \mathbb{N}^n$  with  $|\alpha| \neq |\beta|$ .

Second, let  $\Sigma := \text{Poly}(P) - \text{Poly}(\Lambda)$ . Then, by Lemma 4.9, it will be enough to show that  $\Sigma \cap \mathbb{R}_{\geq 0}^n = \emptyset$ .

We assume otherwise, i.e.  $\Sigma \cap \mathbb{R}_{\geq 0}^n \neq \emptyset$ , and derive a contradiction as follows.

Note first that, by Lemma 4.3, (b), we know that  $\Sigma$  is a rational polytope. By Corollary 4.4, there exists a rational point  $w \in \Sigma \cap \mathbb{R}_{\geq 0}^n$ . Then, by lemma 4.3, (c), there exist  $u \in \text{Poly}(P)$  and  $v \in \text{Poly}(\Lambda)$  such that  $u, v$  are both rational and  $u - v = w \in \Sigma \cap \mathbb{R}_{\geq 0}^n \subset \mathbb{R}_{\geq 0}^n$ . But this contradicts to Lemma 6.4. Hence we have proved the theorem.  $\square$

Finally, let us point out that, by similar arguments as in the proof of Theorem 6.1, it is easy to see that Conjecture 1.2 also holds for the following case.

**Corollary 6.5.** *Let  $\alpha, \beta \in \mathbb{N}^n$  with  $|\alpha| \neq |\beta|$  and  $P(z) = az^\alpha + bz^\beta$  for some  $a, b \in \mathbb{C}$ . Let  $\Lambda = \Lambda(\partial)$  with  $\Lambda(\xi)$  homogeneous. Then Conjecture 1.2 holds for  $\Lambda, P(z)$  and any  $g(z) \in \mathbb{C}[z]$ .*

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